The linear span of uniform matrix product states (uMPS)

joint work with Claudia De Lazzari and Tim Seynnaeve

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Overview

1. Tensor Networks
2. Uniform Matrix Product States (uMPS)
3. Results
4. Open Problem

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What are Tensor Networks?

vector v_j

matrix M_{ij}

3-index tensor T_{ijk}

What are Tensor Networks?

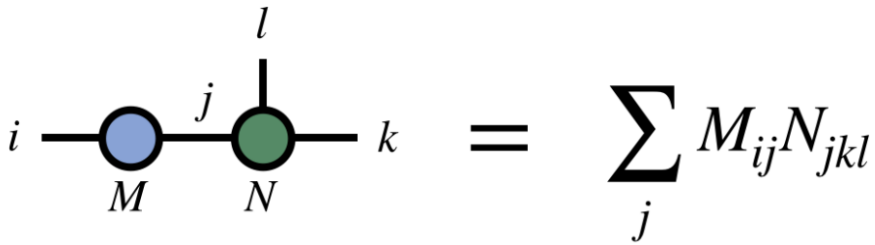

Figure: Diagrammatic representation of Tensor Contraction

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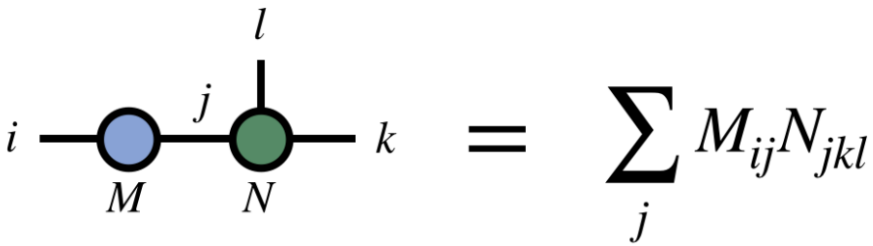

Figure: Diagrammatic representation of Tensor Contraction

- Tensor networks are diagrammatic representation of tensor contraction.

Matrix Operations via Tensor Networks

- It can be thought of as generalization of matrix multiplication to higher order tensors.

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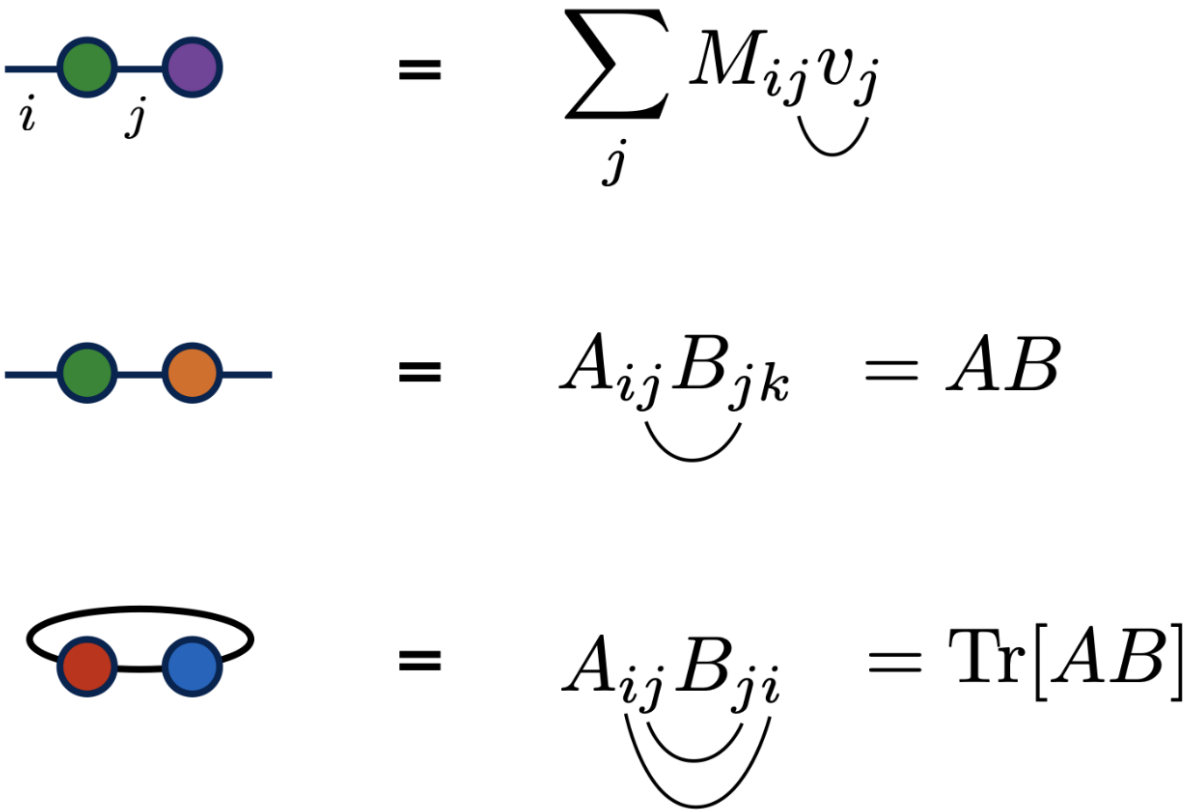

Figure: Matrix operations via Tensor Networks

Motivation from Quantum Physics

- Quantum Many-Body system consisting of d particles, each one with the wave function residing in finite dimensional Hilbert space $\mathcal{H}_i$, with $\dim(\mathcal{H}_i) = n$ and the orthonormal basis of $\mathcal{H}_i$ as $\{|e_{h_i}\rangle\}_{h_i=1}^{n}$.

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- The wave function of many-body system is a tensor product of states $\mathcal{H} = \otimes_{i=1}^{d} \mathcal{H}_i$.

$$|\psi\rangle = \sum_{h_1,\ldots,h_d=1}^{n} \mathcal{A}_{h_1\cdots h_d} |e_{h_1}\rangle \otimes \cdots \otimes |e_{h_d}\rangle$$

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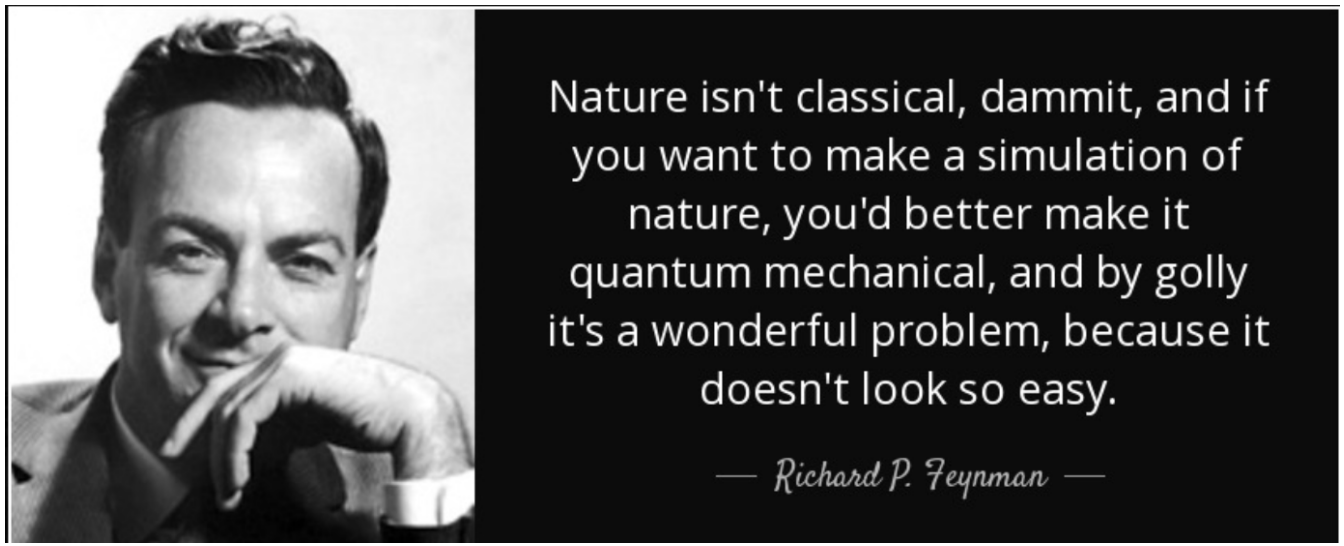

Figure: One solution: Quantum Computers!

Another Solution: Tensor Networks!

- Fortunately, most of the physically relevant states occupies exponentially small volume in many-body Hilbert Space [PQSV11].

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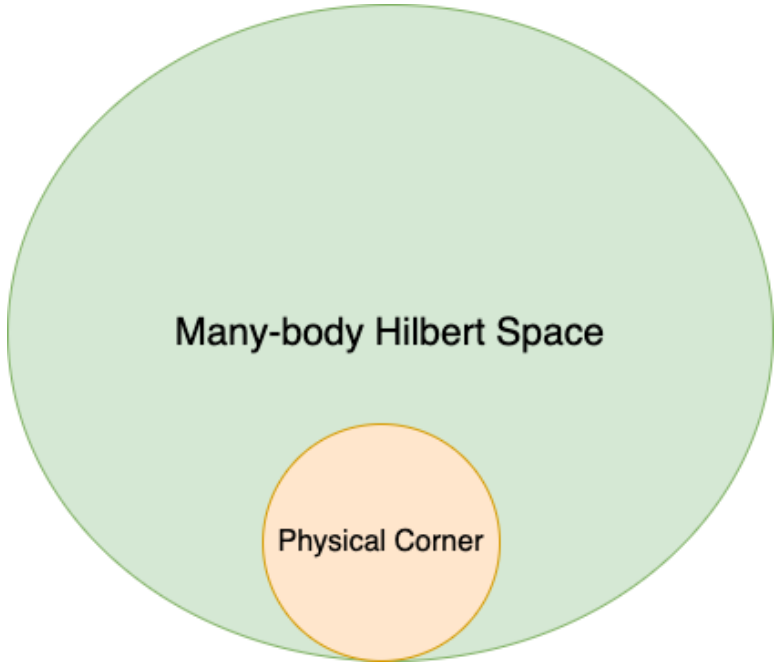

Figure: Physical Corner of many-body Hilbert Space

Uniform Matrix Product States (uMPS)

Definition

The uniform Matrix Product State parametrization is given by the map

$$\phi : (\mathbb{C}^{m\times m})^n \to (\mathbb{C}^n)^{\otimes d}$$

$$(A_0, \ldots, A_{n-1}) \mapsto \sum_{0\leq i_1,\ldots,i_d\leq n-1} \mathrm{Tr}(A_{i_1}\cdots A_{i_d})\; e_{i_1}\otimes\cdots\otimes e_{i_d}.$$

$$\mathrm{uMPS}(m,n,d) = \overline{Im(\phi)}$$

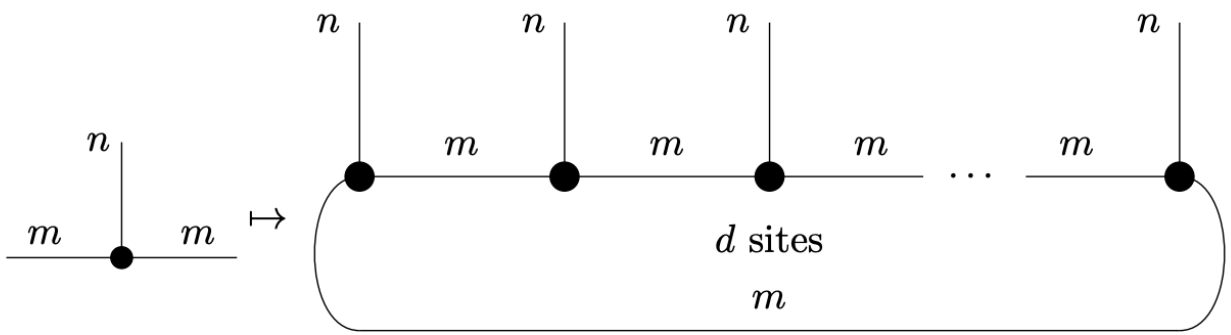

Figure: Graphical representation of uMPS(m,n,d)

Fundamental Questions

- What is the dimension of $\langle \text{uMPS}(m, n, d) \rangle$?

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- What is the dimension of $\langle \text{uMPS}(m, n, d) \rangle$?
- For which parameters $m, n, d \in \mathbb{N}$ does $\langle \text{uMPS}(m, n, d) \rangle$ fill the ambient space?

Observations

$$\mathrm{Cyc}^d(\mathbb{C}^n) := \{\omega \in (\mathbb{C}^n)^{\otimes d} \mid \sigma \cdot \omega = \omega \quad \forall \sigma \in C_d\}.$$

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Observation 1

The set $\mathrm{uMPS}(m, n, d) \subseteq \mathrm{Cyc}^d(\mathbb{C}^n)$. Given $M_1, \ldots, M_d \in \mathbb{C}^{m \times m}$, $\sigma \in C_d$

$$\mathrm{Tr}(M_1 \cdots M_d) = \mathrm{Tr}(M_{\sigma(1)} \cdots M_{\sigma(d)}).$$

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$$\mathrm{Dih}^d(\mathbb{C}^n) := \{\omega \in (\mathbb{C}^n)^{\otimes d} \mid \sigma \cdot \omega = \omega \quad \forall \sigma \in D_{2d}\}.$$

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Observation 2 ([Gre14, Theorem 1.1])

$\mathrm{uMPS}(2,2,d) \subseteq \mathrm{Dih}^d(\mathbb{C}^n)$. Given $A_{i_1}, \cdots, A_{i_d} \in \mathbb{C}^{2\times 2}$

$$\mathrm{Tr}(A_{i_1} \cdots A_{i_d}) = \mathrm{Tr}(A_{i_d} \cdots A_{i_1}).$$

Idea 1 - Cayley-Hamilton Technique

Cayley-Hamilton Technique

Let $c = (c_1, \ldots, c_s) \in \mathbb{C}^s$ be a vector of coefficients and $\{i_\ell^j\}_{1 \leq \ell \leq d, 1 \leq j \leq s}$ be indices; with $i_\ell^j \in [n]$. Assume that for every n-tuple $(A_0, \ldots, A_{n-1})$ of $m \times m$ matrices and every $k < m$ the following identity holds:

$$\sum_{j=1}^{s} c_j \operatorname{Tr}(A_{i_1^j} \cdots A_{i_d^j} A_0^k) = 0.$$

Then the same identity holds for arbitrary $k \in \mathbb{N}$.

Example - 2×2 matrices

The following identity holds for any 2×2 matrices A_0, A_1, A_2, A_3 and $k \geq 0$:

$$\begin{aligned} &\mathrm{Tr}(A_1 A_2 A_0 A_3 A_0^k) + \mathrm{Tr}(A_2 A_3 A_0 A_1 A_0^k) + \mathrm{Tr}(A_3 A_1 A_0 A_2 A_0^k) \\ =\ &\mathrm{Tr}(A_1 A_0 A_2 A_3 A_0^k) + \mathrm{Tr}(A_2 A_0 A_3 A_1 A_0^k) + \mathrm{Tr}(A_3 A_0 A_1 A_2 A_0^k). \end{aligned}$$

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$$\begin{aligned} &\underline{\mathrm{Tr}(A_1A_2A_0A_3)} + \underline{\mathrm{Tr}(A_2A_3A_0A_1)} + \underline{\mathrm{Tr}(A_3A_1A_0A_2)} \\ =&\underline{\mathrm{Tr}(A_1A_0A_2A_3)} + \underline{\mathrm{Tr}(A_2A_0A_3A_1)} + \underline{\mathrm{Tr}(A_3A_0A_1A_2)} \end{aligned}$$

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Strict Containment

Theorem 1

If $n \geq 3$ and $d \geq \frac{(m+1)(m+2)}{2}$, then uMPS$(m, n, d)$ is contained in a proper linear subspace of the space of cyclically invariant tensors.

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Sketch of Proof:

- We need to find a non-trivial linear relation between traces of matrices which is not given by cyclic permutation.
- We proved the following technical identity using Cayley-Hamilton trick

$$\sum_{\sigma \in \mathfrak{S}_m, \tau \in C_{m+1}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \operatorname{Tr}(A_{\tau(0)} B^{\sigma(0)} \cdots A_{\tau(m-1)} B^{\sigma(m-1)} A_{\tau(m)} B^{\ell}) = 0.$$

Idea 2 - Trace Parametrization

A *bracelet* (of length d on the alphabet $[n]$) is an equivalence class of words, where two words are equivalent if they agree up to the action D_{2d}.

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Technical Lemma

For every bracelet $b = (b_1, \ldots, b_k)$, there is a unique polynomial

$$P_b(T_0, T_1, T_{00}, T_{01}, T_{11}) \in \mathbb{C}[T_0, T_1, T_{00}, T_{01}, T_{11}]$$

such that for every pair (A_0, A_1) of 2×2 matrices, the following equality holds:

$$\mathrm{Tr}(A_{b_1} \cdots A_{b_k}) = P_b(\mathrm{Tr}(A_0), \mathrm{Tr}(A_1), \mathrm{Tr}(A_0^2), \mathrm{Tr}(A_0 A_1), \mathrm{Tr}(A_1^2)).$$

Idea 2 - Trace Parametrization

- uMPS$(2, 2, d)$ is the image of the polynomial map

$$\psi : \mathbb{C}^5 \to \mathrm{Dih}^d(\mathbb{C}^2)$$

$$(T_0, T_1, T_{00}, T_{01}, T_{11}) \mapsto \sum_b P_b(T_0, T_1, T_{00}, T_{01}, T_{11}) e_b,$$

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  where b runs over all bracelets of length d.
- This is the trace parametrization of uMPS$(2, 2, d)$.

Bound on dimensions

Theorem 2

For every $d \in \mathbb{N}$, we have the inequality

$$\dim\langle \text{uMPS}(2,2,d)\rangle \leq \begin{cases} \frac{1}{192}(d+6)(d+4)^2(d+2) & \text{for } d \text{ even,} \\ \frac{1}{192}(d+7)(d+5)(d+3)(d+1) & \text{for } d \text{ odd.} \end{cases}$$

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Sketch of Proof:

- Using the trace parametrization of $\text{uMPS}(2,2,d)$, $\dim\langle \text{uMPS}(2,2,d)\rangle$ is at most the number of degree d monomials in $\mathbb{C}[T_0, T_1, T_{00}, T_{01}, T_{11}]$.

Open Problem

Conjecture

$$\dim\langle \text{uMPS}(2,2,d)\rangle = \begin{cases} \frac{1}{192}(d^4 - 4d^2 + 192d + 192) & \text{for } d \text{ even,} \\ \frac{1}{192}(d^4 - 10d^2 + 192d + 201) & \text{for } d \text{ odd.} \end{cases}$$

References

The work presented here is based on:

- De Lazzari C, Motwani HJ, Seynnaeve T. **The linear span of uniform matrix product states.** arXiv preprint arXiv:2204.10363. 2022 Apr 21.
- Code available at: `https://github.com/harshitmotwani2015/uMPS/`

Pictures of Tensor Networks are taken from `https://tensornetwork.org/`

Additional Reference

- Poulin D, Qarry A, Somma R, Verstraete F. Quantum simulation of time-dependent Hamiltonians and the convenient illusion of Hilbert space. Physical review letters. 2011 Apr 29;106(17):170501.
- Greene J. Traces of matrix products. The Electronic Journal of Linear Algebra. 2014 Jan 1;27:716-34.